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SOME PROBLEMS OF THE DYNAMICAL THEORY OF LINEAR NONMAGNETIZABLE PIEZOELECTRIC MICROPOLAR THERMOELASTICITY

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Abstract—In this paper the dynamical theory of linear nonmagnetizable piezoelectric micropolar thermoelasticity is considered. Using a tensorial description, basic concepts such as admissible process, dynamical nonmagnetizable piezoelectric micropolar thermoelastic process and mixed problem are defined. A basic result is proved and, by using it, some reciprocity relations are deduced. The results obtained here allow us to prove both an uniqueness theorem of a solution of the mixed problem and a Green representation integral formula for the same solution. © 1997 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

In this paper we will be interested in the correlations between the entities describing both mechanical and thermal properties of a solid, and those showing its electromagnetic behaviour. Polarizable solid materials may, when deformed, exhibit electrical phenomena. In some types of crystals an application of internal forces as body forces, couple-body forces and heat sources, as well as the presence of forces and a heat flux on their surface produces electric polarization. Usually, the presence of an electric field creates a magnetic one that it interacts with. This phenomenon is called the direct piezoelectric effect (Novozhilov and Yappa, 1981). We can also have the converse piezoelectric effect consising of the appearance of stresses and thermal variations due to an applied electric field in a crystal solid. Such phenomena when they occur are always associated with the anisotropic solids. A piezoelectric field may be produced under small deformations of anisotropic solids. Isotropic solids do not exhibit this property. Because of their importance in engineering applications, the classical theories for these effects have become highly specialized disciplines. A short history and a comprehensive list in this area until the middle of the 20th century may be found in Cady (1946). Any such theory must be based on simultaneous application both of the principles of mechanics and of electromagnetism.

The piezoelectric effects in an anisotropic Hookean body have been known for a long time. Theoretical studies about the phenomenon of piezoelectricity were given at the beginning of the 19th century and since these have been elaborated and many studies in correlation with the theories of continuum mechanics have been developed (see, for example, Voigt, 1887; Maugin, 1988). Many special problems of piezoelectricity have been solved by various authors. We quote here Brzeziński (1978), Nowacki (1978), Ieşan (1989, 1990) and Wang (1992).

Numerous anomalies in the behaviour of some crystals, quartz and diamond and others, in piezoelectric effects, proved the necessity of construction of an appropriate theory. Precisely, the behaviour of piezoelectric crystals shows that the number of material constants in Voigt's theory is not sufficient. Starting from a theory of a hemitropic Cosserat body exhibiting anisotropy, Nowacki (1986) elaborated a unified mathematical theory of the piezoelectricity including thermal and electromagnetic effects, known as the linear theory of piezoelectric micropolar thermoelasticity for inhomogeneous and anisotropic bodies.

Here, we consider the dynamic theory of linear nonmagnetizable piezoelectric micropolar thermoelasticity of the anisotropic and inhomogeneous bodies. A tensorial description is used, as in Gurtin (1972) and Carlson (1972), which, in our opinion, is very convenient to prove many results concerning reciprocity relations, uniqueness, variational principles *et al.*

In Section 2 we give some mathematical preliminaries necessary for a tensorial description of the above mentioned theory. Tensorial functions, inner product, convolution, Hamilton's operator and some properties concerning these notions are presented.

As in a previous paper (Crăciun, 1994), in Section 3 we define the admissible process in the dynamic theory of linear nonmagnetizable piezoelectric micropolar thermoelasticity for an anisotropic and inhomogeneous body occupying a region B of space. The fundamental notion of this theory is that of dynamic nonmagnetizable piezoelectric micropolar thermoelastic process in an inhomogeneous and anisotropic body.

In Section 4, the mixed problem of this theory is defined. Theorem 1 gives equivalent forms of some basic equations governing this theory.

In Section 5, a very important result, based on the papers of Ieşan (1989; 1990), is proved. Note that the result established in Lemma 1 may be used to prove a uniqueness theorem of the solution of the mixed problem (Crăciun, 1994).

Then, this result is successfully used in Section 6 to prove reciprocal theorems. The special cases of inhomogeneous initial conditions is considered. We note that the reciprocity relations established here could be used to give Green representation integral formula for a solution of the mixed problem.

2. SOME MATHEMATICAL PRELIMINARIES

We shall consider an inhomogeneous and anisotropic body that, beginning at time $t = 0$, occupies the regular region B of the Euclidian three-dimensional space whose associated vector space is $\mathbb{R}^3$. The region B is bounded by the piecewise smooth closed surface ∂B. We denote by $\mathbf{n}$ the outward unit normal in a point of ∂B. In a linear theory, the motion of a body is referred to a fixed system of rectangular Cartesian axes Ox_i, whose unit vectors are $\mathbf{e}_i$, $i = 1$–3. The time interval $[0, t_0)$ will be denoted by I. We will meet here functions depending on the position vector $\mathbf{x} = (x_1, x_2, x_3)$ of a point in $\bar{B} = B \cup \partial B$ and on time t in I, whose domains of definition may be the Cartesian product $\bar{B} \times I$ or one of the sets $B \times (0, t_0)$ and $\partial B \times I$. The values $f(\mathbf{x}, t)$ of a function f may be in the set of real numbers $\mathbb{R}$ or in the linear space τ_p of all p-tensors, where $\tau_p = \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \ldots \otimes \mathbb{R}^3$, p-times, and $\otimes$ stands for the tensorial product. It is also possible to have functions depending only on $\mathbf{x}$. Therefore, in this paper we will use both real (scalar) and tensorial functions. A tensorial function having the values in τ_p will also be named a p-order tensor. We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers 1–3; the summation over repeated indices is implied; subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate; a superimposed dot designates partial differentiation with respect to t; and a centred dot between two p-order tensors shows that the inner product in τ_p is effected.

For a tensorial function with values in τ_p we write $f_{i_1 i_2 \ldots i_p}$ for its components. The inner product of the two p-order $f(\mathbf{x}, t)$ and $g(\mathbf{x}, t)$ is given by

$$f(\mathbf{x}, t) \cdot g(\mathbf{x}, t) = f_{i_1 i_2 \ldots i_p}(\mathbf{x}, t) g_{i_1 i_2 \ldots i_p}(\mathbf{x}, t). \tag{1}$$

The convolution of two p-order tensors f and g is the function $f * g : \bar{B} \times I \to \mathbb{R}$, defined by:

$$(f * g)(\mathbf{x}, t) = \int_0^t f(\mathbf{x}, t-\tau) \cdot g(\mathbf{x}, \tau) \, d\tau. \tag{2}$$

It is possible that in the place of f is a scalar function, i or 1, defined in I by:

$$i(t) = t, \quad 1(t) = 1. \tag{3}$$

We will write $\bar{h}$ for $1 * h$, that is:

$$\bar{h}(\mathbf{x}, t) = (1 * h)(\mathbf{x}, t) = \int_0^t h(\mathbf{x}, \tau)\,\mathrm{d}\tau. \tag{4}$$

Obviously, from eqn (4) it follows:

$$\dot{\bar{h}}(\mathbf{x}, t) = h(\mathbf{x}, t) \quad \text{and} \quad h(\mathbf{x}, t) - h(\mathbf{x}, 0) = (1 * \dot{h})(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \bar{B} \times I. \tag{5}$$

It is also possible that f and g are vectors in $\mathbb{R}^n$ vector space in which the inner product is the standard one. A scalar or tensorial function is in the function class $C^{p,q}$ on $\bar{B} \times I$ or on $B \times (0, t_0)$ or on $\partial B \times I$, if all of the functions $f^{(n)}_{,i_1 i_2 \ldots i_m}$, $m = 1,2,\ldots,p$, $n = 1,2,\ldots,q$ exist and are continuous on their domains of definition.

By ∇ we understand the Hamilton operator whose expression is given by:

$$\nabla = \mathbf{e}_i \frac{\partial}{\partial x_i}, \tag{6}$$

and by $\nabla()$, $\nabla\cdot()$ and $\nabla \times ()$ we mean, respectively, the gradient, divergence and curl of a field founded between parentheses. If f is a two order tensor, g and h are tensors of one order, that is vectors, and q is a scalar function, then from the properties of the ∇-operator given by eqn (6), we have the following identities (Gurtin, 1972):

$$\begin{aligned} f\cdot(\nabla g)^T &= \nabla\cdot(f[g]) - (\nabla\cdot f^T)\cdot g, \\ (\nabla\cdot h)q &= \nabla\cdot(qh) - h\cdot(\nabla q), \\ \nabla\cdot(g \times h) &= h\cdot(\nabla \times g) - g\cdot(\nabla \times h), \end{aligned} \tag{7}$$

where $f[g]$ and $g \times h$ (the vector product) are the vectors, respectively, given by:

$$f[g] = f_{ij} g_j \mathbf{e}_i, \quad g \times h = \varepsilon_{ijk} g_j h_k \mathbf{e}_i, \tag{8}$$

here ε_{ijk} being alternating symbols.

A p-order tensor will be written in the following as a letter in boldface, letters in italic stand for scalar functions and a capital italic letter, with the exception of L, is the notation for a vectorial function with values in $\mathbb{R}^{10}$ vector space.

3. DEFINITIONS AND GOVERNING EQUATIONS

Definition 1

By an admissible process in B in the dynamical theory of linear nonmagnetizable piezoelectric micropolar thermoelasticity we mean an ordeal array of functions $\varrho = (\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\theta}, \mathbf{e}, \mathbf{b}, \mathbf{E}, \boldsymbol{\kappa}, \mathbf{S}, \mathbf{M}, \boldsymbol{S}, \mathbf{q}, \mathbf{d})$, with the following properties:

$-\mathbf{u}, \boldsymbol{\varphi} : \bar{B} \times I \to \tau_1, \mathbf{u}, \boldsymbol{\varphi} \in C^{2,2}(B \times (0, t_0)), \mathbf{u}, \dot{\mathbf{u}}, \ddot{\mathbf{u}}, \boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}, \ddot{\boldsymbol{\varphi}}, \nabla\mathbf{u}, \nabla\dot{\mathbf{u}}, \nabla\boldsymbol{\varphi}, \nabla\dot{\boldsymbol{\varphi}} \in C^{0,0}(\bar{B} \times I)$;

$-\boldsymbol{\theta}, \boldsymbol{S} : \bar{B} \times I \to \mathbb{R}, \boldsymbol{\theta} \in C^{2,1}(B \times (0, t_0)), \boldsymbol{S} \in C^{0,1}(B \times (0, t_0)), \boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \nabla\boldsymbol{\theta}, \boldsymbol{S}, \dot{\boldsymbol{S}} \in C^{0,0}(\bar{B} \times I)$;

$-\mathbf{E}, \boldsymbol{\kappa} : \bar{B} \times I \to \tau_2, \mathbf{E}, \boldsymbol{\kappa} \in C^{1,1}(B \times (0, t_0)), \mathbf{E}, \dot{\mathbf{E}}, \boldsymbol{\kappa}, \dot{\boldsymbol{\kappa}} \in C^{0,0}(\bar{B} \times I)$;

$-\mathbf{e}, \mathbf{g}, \mathbf{b} : \bar{B} \times I \to \tau_1, \mathbf{g} \in C^{1,0}(B \times (0, t_0)), \mathbf{e}, \mathbf{b} \in C^{1,1}(B \times (0, t_0)), \mathbf{g}, \mathbf{e}, \nabla \times \mathbf{e}, \nabla \times \mathbf{b} \in C^{0,0}(\bar{B} \times I)$;

$-\mathbf{S}, \mathbf{M} : \bar{B} \times I \to \tau_2, \mathbf{S}, \mathbf{M} \in C^{1,0}(B \times (0, t_0)), \mathbf{S}, \mathbf{M}, \nabla\cdot\mathbf{S}, \nabla\cdot\mathbf{M} \in C^{0,0}(\bar{B} \times I)$;

$-\mathbf{q}, \mathbf{d} : \bar{B} \times I \to \tau_1, \mathbf{q} \in C^{1,0}(B \times (0, t_0)), \mathbf{d} \in C^{1,1}(B \times (0, t_0)), \mathbf{q}, \mathbf{d}, \nabla\cdot\mathbf{q} \in C^{0,0}(\bar{B} \times I)$.

In the definition above, $\mathbf{u}$ and $\boldsymbol{\varphi}$ are the displacement and rotation vectors, respectively, $\boldsymbol{\theta}$ is the difference temperature measured from the constant absolute temperature T_0, $\mathbf{e}$ is the electric field, $\mathbf{b}$ is the magnetic flux densiy, $\mathbf{E}$ and $\boldsymbol{\kappa}$ are measures of deformation, $\mathbf{g}$ is the temperature gradient, $\mathbf{S}$, $\mathbf{M}$ and S are, respectively, the stress tensor, couple-stress tensor and entropy, $\mathbf{q}$ is the heat flux vector and $\mathbf{d}$ is the electric displacement vector field.

The components of ϱ need not be related and the set A of all admissible processes in B is a real vector space provided the addition of two admissible processes and the multiplication by a scalar of an admissible process are defined in a natural manner.

Definition 2

An external system of causes in B is the ordered array both of real and vectorial functions of the form $\mathfrak{F} = (\mathbf{X}, \mathbf{Y}, r, \mathbf{j}, \mathbf{s}, \mathbf{m}, q, \tilde{\mathbf{e}}, \tilde{\mathbf{b}})$ satisfying the conditions:

—$\mathbf{X}, \mathbf{Y}, \mathbf{j} : \bar{B} \times I \to \tau_1$, $r : \bar{B} \times I \to \mathbb{R}$, all continuous on $\bar{B} \times I$;
—$\mathbf{s}, \mathbf{m}, \tilde{\mathbf{e}}, \tilde{\mathbf{b}} : \partial B \times I \to \tau_1$, $q : \partial B \times I \to \mathbb{R}$, all continuous on $\partial B \times I$.

Here, $\mathbf{X}, \mathbf{Y}$ and $\mathbf{j}$ are, respectively, the body force vector, couple-body force vector and electric current density vector, r is the heat source, $\mathbf{s}$ and $\mathbf{m}$ are the surface traction and couple-stress vectors, respectively, possible to be piecewise regular functions, but continuous in time, and $\tilde{\mathbf{e}}$, $\tilde{\mathbf{b}}$ are given functions which will be related to $\mathbf{e}$ and $\mathbf{b}$.

Definition 3

By a dynamical nonmagnetizable piezoelectric micropolar thermoelastic process in B corresponding to the external system of causes $\mathfrak{F}$ we mean an admissible process $\varrho \in A$ that satisfies the following relations and equations:

—the geometrical relations:

$$\mathbf{E} = (\nabla \mathbf{u})^T - \boldsymbol{\varphi} \mathbf{L}, \tag{9}$$

$$\boldsymbol{\kappa} = (\nabla \boldsymbol{\varphi})^T, \tag{10}$$

$$\mathbf{g} = \nabla \boldsymbol{\theta}, \quad \text{on } \bar{B} \times I; \tag{11}$$

—the constitutive equations:

$$\mathbf{S} = \mathbf{A}[\mathbf{E}] + \mathbf{B}[\boldsymbol{\kappa}] - \boldsymbol{\theta}\boldsymbol{\eta} - \mathbf{e}\mathfrak{C}, \tag{12}$$

$$\mathbf{M} = \mathbf{E}\mathbf{B} + \mathbf{C}[\boldsymbol{\kappa}] - \boldsymbol{\theta}\boldsymbol{\zeta} - \mathbf{e}\mathbf{D}, \tag{13}$$

$$\mathbf{d} = \mathfrak{C}[\mathbf{E}] + \mathbf{D}[\boldsymbol{\kappa}] + \boldsymbol{\theta}\mathbf{G} + \boldsymbol{\beta}[\mathbf{e}], \tag{14}$$

$$S = \boldsymbol{\eta} \cdot \mathbf{E} + \boldsymbol{\zeta} \cdot \boldsymbol{\kappa} + \frac{c_\varepsilon}{T_0}\boldsymbol{\theta} + \mathbf{e} \cdot \mathbf{G}, \quad \text{on } \bar{B} \times I; \tag{15}$$

—the Fourier's law:

$$\mathbf{q} = -\mathbf{K}[\mathbf{g}], \quad \text{on } \bar{B} \times I; \tag{16}$$

—the motion equations:

$$\nabla \cdot \mathbf{S}^T + \mathbf{X} = \rho \ddot{\mathbf{u}}, \tag{17}$$

$$\nabla \cdot \mathbf{M}^T + \mathbf{L}[\mathbf{S}] + \mathbf{Y} = \mathbf{J}[\ddot{\boldsymbol{\varphi}}], \quad \text{on } B \times (0, t_0); \tag{18}$$

—the energy equation:

$$T_0\dot{S}+\nabla\cdot\mathbf{q}-r = 0, \quad \text{on } B\times(0,t_0); \tag{19}$$

—the Maxwell equations (in Gaussian system of units):

$$c\nabla\times\mathbf{e}+\dot{\mathbf{b}} = 0, \tag{20}$$

$$c\nabla\times\mathbf{b}-\dot{\mathbf{d}} = \mathbf{j}, \quad \text{on } B\times(0,t_0). \tag{21}$$

Equations $\nabla\cdot\mathbf{b} = 0$ and $\nabla\cdot\mathbf{d} = 0$ will not be considered here because they are consequences of eqns (20) and (21), of the solenoidal initial conditions and of the local equation of conservation of electric charge.

In these relations and equations appear new functions and constants with the following significances and properties:
—$\mathbf{A}, \mathbf{B}, \mathbf{C}: \bar{B}\to\tau_4$ are continuously differentiable functions on $\bar{B}$ satisfying the symmetry relations:

$$\mathbf{A}[\mathbf{U}]\cdot\mathbf{V} = \mathbf{U}\cdot\mathbf{A}[\mathbf{V}], \quad \mathbf{C}[\mathbf{U}]\cdot\mathbf{V} = \mathbf{U}\cdot\mathbf{C}[\mathbf{V}], \quad \forall\,\mathbf{U},\mathbf{V}\in\tau_2; \tag{22}$$

—$\boldsymbol{\eta}, \boldsymbol{\zeta}, \boldsymbol{\beta}, \mathbf{K}, \mathbf{J}: \bar{B}\to\tau_2$ are continuously differentiable functions on $\bar{B}$ having the symmetry properties:

$$\boldsymbol{\beta}[\mathbf{u}]\cdot\mathbf{v} = \mathbf{u}\cdot\boldsymbol{\beta}[\mathbf{v}], \quad \mathbf{K}[\mathbf{u}]\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{K}[\mathbf{v}], \quad \mathbf{J}[\mathbf{u}]\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{J}[\mathbf{v}], \quad \forall\,\mathbf{u},\mathbf{v}\in\tau_1; \tag{23}$$

—$\mathfrak{C}, \mathbf{D}: \bar{B}\to\tau_3$ and $\mathbf{G}: \bar{B}\to\tau_1$ are continuous differentiable functions on $\bar{B}$;

—$\mathbf{L} = \varepsilon_{ijk}\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k$, ρ is a strictly positive continuous real function on $\bar{B}$ (the material density), ε and c are, respectively, the specific heat of the body and velocity of light in vacuum;

—$\boldsymbol{\varphi}\mathbf{L}, \mathbf{eD}, \mathbf{EB}$ and $\mathbf{e}\mathfrak{C}$ are conventions of notation for the following tensors:

$$\boldsymbol{\varphi}\mathbf{L} = \varphi_i\varepsilon_{ijk}\mathbf{e}_j\otimes\mathbf{e}_k, \quad \mathbf{eD} = E_i d_{ijk}\mathbf{e}_j\otimes\mathbf{e}_k, \quad \mathbf{EB} = \gamma_{ij}B_{ijks}\mathbf{e}_k\otimes\mathbf{e}_s, \quad \mathbf{e}\mathfrak{C} = E_i e_{ijk}\mathbf{e}_j\otimes\mathbf{e}_k. \tag{24}$$

All the assertions that follow eqn (21) are known as material properties of the body B. The functions $\mathbf{s}$, $\mathbf{m}$, q, $\tilde{\mathbf{e}}$, $\tilde{\mathbf{b}}$ depend on $\mathbf{n}$ as follows:

$$\mathbf{s} = \mathbf{S}^T[\mathbf{n}], \quad \mathbf{m} = \mathbf{M}^T[\mathbf{n}], \quad q = \mathbf{q}\cdot\mathbf{n}, \quad \tilde{\mathbf{e}} = \mathbf{e}\times\mathbf{n}, \quad \tilde{\mathbf{b}} = \mathbf{b}\times\mathbf{n}. \tag{25}$$

4. MIXED PROBLEMS

In this section, in addition to the specification of the body and its material properties, we assume that the following data are given:

—$\mathfrak{F}$ is an external system of causes in B;
—the initial displacement, velocity, rotation, angular rotation, entropy, electric displacement and magnetic flux density fields $\mathbf{u}_0$, $\mathbf{v}_0$, $\boldsymbol{\varphi}_0$, $\mathbf{w}_0$, S_0, $\mathbf{d}_0$ and $\mathbf{b}_0$ all continuous on $\bar{B}$;
—$\mathbf{d}_0$ and $\mathbf{b}_0$ solenoidal vector fields on $\bar{B}$;
—the surface dislacement $\hat{\mathbf{u}}$ continuous on $\bar{\Sigma}_1\times I$;
—the surface traction $\hat{\mathbf{s}}$ piecewise regular on $\Sigma_2\times I$ and continuous in time;
—the surface rotation $\hat{\boldsymbol{\varphi}}$ continuous on $\bar{\Sigma}_3\times I$;
—the surface couple-stress vector $\hat{\mathbf{m}}$ piecewise regular on $\Sigma_4\times I$ and continuous in time;
—the surface temperature $\hat{\boldsymbol{\theta}}$ continuous on $\bar{\Sigma}_5\times I$;
—the surface heat flux $\hat{q}$ piecewise regular on $\Sigma_6\times I$ and continuous in time;

—the function $\hat{\tilde{\mathbf{e}}}$ piecewise regular on $\bar{\Sigma}_7 \times I$ and continuous in time;
—the function $\hat{\tilde{\mathbf{b}}}$ piecewise regular on $\Sigma_8 \times I$ and continuous in time.

The sets $\bar{\Sigma}_p$, Σ_{p+1}, $p \in \{1, 3, 5, 7\}$, are dual systems of complementary regular subsets of ∂B with the properties: $\bar{\Sigma}_p \cup \Sigma_{p+1} = \partial B$, $\Sigma_p \cap \Sigma_{p+1} = \phi$. Moreover, any one of these sets may be a piecewise regular surface and also any one of them could be the empty set.

Definition 4

A mixed problem of the dynamical theory of linear nonmagnetizable piezoelectric micropolar thermoelasticity is the problem of finding a dynamical nonmagnetizable piezoelectric micropolar thermoelastic process ϱ in B corresponding to the external system of causes $\mathfrak{F}$ in B satisfying:

—the initial conditions:

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \dot{\mathbf{u}}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}), \quad \boldsymbol{\varphi}(\mathbf{x},0) = \boldsymbol{\varphi}_0(\mathbf{x}), \quad \dot{\boldsymbol{\varphi}}(\mathbf{x},0) = \mathbf{w}_0(\mathbf{x}),$$
$$S(\mathbf{x},0) = S_0(\mathbf{x}), \quad \mathbf{d}(\mathbf{x},0) = \mathbf{d}_0(\mathbf{x}), \quad \mathbf{b}(\mathbf{x},0) = \mathbf{b}_0(\mathbf{x}), \quad \mathbf{x} \in \bar{B}; \tag{26}$$

and

—the boundary conditions:

$$\mathbf{u}(\mathbf{x},t) = \hat{\mathbf{u}}(\mathbf{x},t), \quad (\mathbf{x},t) \in \bar{\Sigma}_1 \times I; \quad \mathbf{s}(\mathbf{x},t) = \hat{\mathbf{s}}(\mathbf{x},t), \quad (\mathbf{x},t) \in \Sigma_2 \times I;$$
$$\boldsymbol{\varphi}(\mathbf{x},t) = \hat{\boldsymbol{\varphi}}(\mathbf{x},t), \quad (\mathbf{x},t) \in \bar{\Sigma}_3 \times I; \quad \mathbf{m}(\mathbf{x},t) = \hat{\mathbf{m}}(\mathbf{x},t), \quad (\mathbf{x},t) \in \Sigma_4 \times I;$$
$$\boldsymbol{\theta}(\mathbf{x},t) = \hat{\boldsymbol{\theta}}(\mathbf{x},t), \quad (\mathbf{x},t) \in \bar{\Sigma}_5 \times I; \quad q(\mathbf{x},t) = \hat{\mathbf{q}}(\mathbf{x},t), \quad (\mathbf{x},t) \in \Sigma_6 \times I;$$
$$\tilde{\mathbf{e}}(\mathbf{x},t) = \hat{\tilde{\mathbf{e}}}(\mathbf{x},t), \quad (\mathbf{x},t) \in \bar{\Sigma}_7 \times I; \quad \tilde{\mathbf{b}}(\mathbf{x},t) = \hat{\tilde{\mathbf{b}}}(\mathbf{x},t), \quad (\mathbf{x},t) \in \Sigma_8 \times \mathrm{I}. \tag{27}$$

Definition 5

We call a solution of the mixed problem a dynamical nonmagnetizable piezoelectric micropolar thermoelastic process in B, if it exists, satisfying eqns (26) and (27).

Note that numerous special cases of the mixed problem above defined can arise when one or more of the subsets $\Sigma_1, \Sigma_2, \ldots, \Sigma_8$ is empty. More complicated boundary-initial-value problems of this theory could also be defined.

Definition 6

An external data system in the dynamical theory of linear nonmagnetizable piezoelectric micropolar thermoelasticity is an ordered array of functions of the form:

$$L = (F, \hat{\mathbf{u}}, \hat{\mathbf{s}}, \hat{\boldsymbol{\varphi}}, \hat{\mathbf{m}}, \hat{\boldsymbol{\theta}}, \hat{q}, \hat{\tilde{\mathbf{e}}}, \hat{\tilde{\mathbf{b}}}, \mathbf{u}_0, \mathbf{v}_0, \boldsymbol{\varphi}_0, \mathbf{w}_0, S_0, \mathbf{b}_0), \tag{28}$$

where

$$F = (\mathbf{X}, \mathbf{Y}, -W, \tilde{\mathbf{j}} - \mathbf{d}_0), \quad W = \frac{1}{T_0} \bar{r} + S_0. \tag{29}$$

Sometimes we will say that $\varrho \in A$ is a solution of the mixed problem in B corresponding to the external data system L.

Theorem 1

An admissible process $\varrho \in A$ that satisfies eqns (9)–(25) and (27) is a solution of the mixed problem in B corresponding to the external data system L if and only if:

$$i * \nabla \cdot \mathbf{S}^T + \bar{\mathbf{X}} = \rho \mathbf{u}, \tag{30}$$

$$i * (\nabla \cdot \mathbf{M}^T + \mathbf{L}[\mathbf{S}]) + \bar{\mathbf{Y}} = \mathbf{J}[\boldsymbol{\varphi}], \tag{31}$$

$$S+\frac{1}{T_0}\nabla\cdot\bar{\mathbf{q}} = W, \tag{32}$$

$$c\nabla\times\bar{\mathbf{e}}+\mathbf{b} = \mathbf{b}_0, \tag{33}$$

$$c\nabla\times\bar{\mathbf{b}}-\mathbf{d} = \bar{\mathbf{j}}-\mathbf{d}_0, \quad \text{on } B\times[0,t_0), \tag{34}$$

where

$$\tilde{\mathbf{X}} = i*\mathbf{X}+\rho(\mathbf{u}_0+t\mathbf{v}_0), \quad \tilde{\mathbf{Y}} = i*\mathbf{Y}+\mathbf{J}[\boldsymbol{\varphi}_0+t\mathbf{w}_0). \tag{35}$$

Proof

To get eqn (30) it must convolute with i of eqn (17) and then use the properties of convolution (Gurtin, 1972). In the same way eqns (31)–(34) can be proved.

5. A BASIC RESULT

If $\varrho\in A$ is a solution of the mixed problem in B corresponding to the external data system L in eqn (28), it is convenable to denote by T and U the following vectors in $\mathbb{R}^{10}$.

$$T=(\mathbf{s},\mathbf{m},\frac{1}{T_0}\bar{q},c\bar{\mathbf{b}}\times\mathbf{n})\,; \quad U=(\mathbf{u},\boldsymbol{\varphi},\boldsymbol{\theta},\mathbf{e}). \tag{36}$$

The inner product of T and U is:

$$T\cdot U = \mathbf{s}\cdot\mathbf{m}+\mathbf{m}\cdot\boldsymbol{\varphi}+\frac{1}{T_0}\bar{q}\boldsymbol{\theta}+c(\bar{\mathbf{b}}\times\mathbf{n})\cdot\mathbf{e}, \tag{37}$$

while the inner product of functions F and U is defined as:

$$F\cdot U = \mathbf{X}\cdot\mathbf{u}+\mathbf{Y}\cdot\boldsymbol{\varphi}-W\boldsymbol{\theta}+(\bar{\mathbf{j}}-\mathbf{d}_0)\cdot\mathbf{e}. \tag{38}$$

Following eqns (2), (37) and (38), convolutions of T by U and of F by U are given, respectively, by:

$$(T*U)(\mathbf{x},t) = \int_0^t T(\mathbf{x},t-\tau)\cdot U(\mathbf{x},\tau)\,\mathrm{d}\tau, \quad (F*U)(\mathbf{x},t) = \int_0^t F(\mathbf{x},t-\tau)\cdot U(\mathbf{x},\tau)\,\mathrm{d}\tau. \tag{39}$$

Lemma 1

If $\varrho^{(\alpha)}$ is a solution of the mixed problem in B corresponding to the external data system $L^{(\alpha)}$, $\alpha = 1, 2$, then

$$z^{\alpha\beta}(\tau,p) = z^{\beta\alpha}(p,\tau), \quad (\tau,p)\in I\times I, \quad \beta = 1,2, \tag{40}$$

where

$$z^{\alpha\beta}(\tau,p) = \int_B F^{(\alpha)}(\mathbf{x},\tau)\cdot U^{(\beta)}(\mathbf{x},p)\,\mathrm{d}v+\int_{\partial B} T^{(\alpha)}(\mathbf{x},\tau)\cdot U^{(\beta)}(\mathbf{x},p)\,\mathrm{d}a$$
$$-\int_B (\rho(\mathbf{x})\ddot{\mathbf{u}}^{(\alpha)}(\mathbf{x},\tau)\cdot\mathbf{u}^{(\beta)}(\mathbf{x},p)+\mathbf{J}(\mathbf{x})[\ddot{\boldsymbol{\varphi}}^{(\alpha)}(\mathbf{x},\tau)]\cdot\boldsymbol{\varphi}^{(\beta)}(\mathbf{x},p))\,\mathrm{d}v$$

$$-\int_B \left(\frac{1}{T_0}\bar{\mathbf{q}}^{(\alpha)}(\mathbf{x},\tau)\cdot\mathbf{g}^{(\beta)}(\mathbf{x},p)-\bar{\mathbf{b}}^{(\alpha)}(\mathbf{x},\tau)\cdot\dot{\mathbf{b}}^{(\beta)}(\mathbf{x},p)\right)\mathrm{d}v. \tag{41}$$

Proof

We introduce the functions $h^{\alpha\beta}:\bar{B}\times I\times I\to\mathbb{R}$, defined by:

$$h^{\alpha\beta}(\mathbf{x},\tau,p)=\mathbf{S}^{(\alpha)}(\mathbf{x},\tau)\cdot\mathbf{E}^{(\beta)}(\mathbf{x},p)+\mathbf{M}^{(\alpha)}(\mathbf{x},\tau)\cdot\boldsymbol{\kappa}^{(\beta)}(\mathbf{x},p)$$
$$-\mathcal{S}^{(\alpha)}(\mathbf{x},\tau)\theta^{(\beta)}(\mathbf{x},p)-\mathbf{d}^{(\alpha)}(\mathbf{x},\tau)\cdot\mathbf{e}^{(\beta)}(\mathbf{x},p). \tag{42}$$

By using geometrical relations (9)–(11), constitutive eqns (12)–(15), the properties of inner product defined by eqn (1), symmetry conditions (22) and (23), and expressions (24) of $\boldsymbol{\varphi}\mathbf{L}$, $\mathbf{eD}$, $\mathbf{EB}$ and $\mathbf{e}\mathfrak{C}$, we can prove that (Crăciun, 1994):

$$h^{\alpha\beta}(\mathbf{x},\tau,p)=h^{\beta\alpha}(\mathbf{x},p,\tau),\quad \text{for any } (\mathbf{x},\tau,p)\in\bar{B}\times I\times I. \tag{43}$$

Let us find a more simple expression for the functions $h^{\alpha\beta}$. For convenience, we will suppress the argument $\mathbf{x}$ in $h^{\alpha\beta}$. Introduction of eqns (9)–(11) in eqn (42), and then the use of equations (32) and (34), properties (7) of ∇-operator, the identity $\mathbf{S}\cdot(\boldsymbol{\varphi}\mathbf{L})=\mathbf{L}[\mathbf{S}]\cdot\boldsymbol{\varphi}$, motion equations (17) and (18), notations (29) and (36), and inner products (37) and (38), leads to:

$$h^{\alpha\beta}(\tau,p)=\nabla\cdot(\mathbf{S}^{(\alpha)}(\tau)[\mathbf{u}^{(\beta)}(p)]+\mathbf{M}^{(\alpha)}(\tau)[\boldsymbol{\varphi}^{(\beta)}(p)]$$
$$+\frac{1}{T_0}\bar{\mathbf{q}}^{(\alpha)}(\tau)\theta^{(\beta)}(p)-c\bar{\mathbf{b}}^{(\alpha)}(\tau)\times\mathbf{e}^{(\beta)}(p)+F^{(\alpha)}(\tau)\cdot U^{(\beta)}(p)$$
$$-\rho\ddot{\mathbf{u}}^{(\alpha)}(\tau)\cdot\mathbf{u}^{(\beta)}(p)-\mathbf{J}[\ddot{\boldsymbol{\varphi}}^{(\alpha)}(\tau)]\cdot\boldsymbol{\varphi}^{(\beta)}(p)-\frac{1}{T_0}\bar{\mathbf{q}}^{(\alpha)}(\tau)\cdot\mathbf{g}^{(\beta)}(p)+\bar{\mathbf{b}}^{(\alpha)}(\tau)\cdot\dot{\mathbf{b}}^{(\beta)}(p). \tag{44}$$

Integration over B of identity (43), where $h^{\alpha\beta}$ has expression (44), then the use of the divergence theorem, relations (25), (36) and (37), proves this lemma.

6. RECIPROCITY RELATIONS

Lemma 1 is very important to establish reciprocity relations in the dynamical theory of linear nonmagnetizable piezoelectric micropolar thermoelasticity for an inhomogeneous and anisotropic body B. First, we have:

Theorem 2

If $\varrho^{(\alpha)}$ is a solution of the mixed problem in B corresponding to the external data system $L^{(\alpha)}$, $\alpha=1,2$, then:

$$\int_B((F^{(1)}*U^{(2)})(\mathbf{x},t)-\bar{\mathbf{b}}^{(1)}(\mathbf{x},t)\cdot\mathbf{b}_0^{(2)}(\mathbf{x}))\,\mathrm{d}v+\int_{\partial B}(T^{(1)}*U^{(2)})(\mathbf{x},t)\,\mathrm{d}a$$
$$-\int_B\rho(\mathbf{x})(\dot{\mathbf{u}}^{(1)}(\mathbf{x},t)\cdot\mathbf{u}_0^{(2)}(\mathbf{x})-\mathbf{v}_0^{(1)}(\mathbf{x})\cdot\mathbf{u}^{(2)}(\mathbf{x},t))\,\mathrm{d}v$$
$$-\int_B(\mathbf{J}(\mathbf{x})[\dot{\boldsymbol{\varphi}}^{(1)}(\mathbf{x},t)]\cdot\boldsymbol{\varphi}_0^{(2)}(\mathbf{x})-\mathbf{J}(\mathbf{x})[\mathbf{w}_0^{(1)}(\mathbf{x})]\cdot\boldsymbol{\varphi}^{(2)}(\mathbf{x},t))\,\mathrm{d}v$$
$$=\int_B((F^{(2)}*U^{(1)})(\mathbf{x},t)-\bar{\mathbf{b}}^{(2)}(\mathbf{x},t)\cdot\mathbf{b}_0^{(1)}(\mathbf{x}))\,\mathrm{d}v+\int_{\partial B}(T^{(2)}*U^{(1)})(\mathbf{x},t)\,\mathrm{d}a$$

$$-\int_B \rho(\mathbf{x})(\dot{\mathbf{u}}^{(2)}(\mathbf{x},t)\cdot\mathbf{u}_0^{(1)}(\mathbf{x})-\mathbf{v}_0^{(2)}(\mathbf{x})\cdot\mathbf{u}^{(1)}(\mathbf{x},t))\,\mathrm{d}v$$

$$-\int_B (\mathbf{J}(\mathbf{x})[\dot{\boldsymbol{\varphi}}^{(2)}(\mathbf{x},t)]\cdot\boldsymbol{\varphi}_0^{(1)}(\mathbf{x})-\mathbf{J}(\mathbf{x})[\mathbf{w}_0^{(2)}(\mathbf{x})]\cdot\boldsymbol{\varphi}^{(1)}(\mathbf{x},t))\,\mathrm{d}v. \tag{45}$$

Proof

In identity (40) we take $\alpha = 1$, $\beta = 2$, $p = t-\tau$ and then integrate with respect to τ from 0 to t. Taking into account eqns (2) and (39) we obtain:

$$\int_B (F^{(1)} * U^{(2)}+\bar{\mathbf{b}}^{(1)} * \dot{\mathbf{b}}^{(2)}-\frac{1}{T_0}\bar{\mathbf{q}}^{(1)} * \mathbf{g}^{(2)}-\rho\ddot{\mathbf{u}}^{(1)} * \mathbf{u}^{(2)}-\mathbf{J}[\ddot{\boldsymbol{\varphi}}^{(1)}] * \boldsymbol{\varphi}^{(2)})(\mathbf{x},t)\,\mathrm{d}v$$

$$+\int_{\partial B}(T^{(1)} * U^{(2)})(\mathbf{x},t)\,\mathrm{d}a$$

$$=\int_B (F^{(2)} * U^{(1)}+\bar{\mathbf{b}}^{(2)} * \dot{\mathbf{b}}^{(1)}-\frac{1}{T_0}\bar{\mathbf{q}}^{(2)} * \mathbf{g}^{(1)}-\rho\ddot{\mathbf{u}}^{(2)} * \mathbf{u}^{(1)}-\mathbf{J}[\ddot{\boldsymbol{\varphi}}^{(2)}] * \boldsymbol{\varphi}^{(1)})(\mathbf{x},t)\,\mathrm{d}v$$

$$+\int_{\partial B}(T^{(2)} * U^{(1)})(\mathbf{x},t)\,\mathrm{d}a. \tag{46}$$

Fourier's law (16), the symmetry of tensor **K**, relations (2) and (4), and a property of inner product (1) give:

$$\bar{\mathbf{q}}^{(1)} * \mathbf{g}^{(2)} = \bar{\mathbf{q}}^{(2)} * \mathbf{g}^{(1)}. \tag{47}$$

From eqns (2), (4) and (5), and properties of convolution (Gurtin, 1972) we get:

$$(\ddot{\mathbf{u}}^{(1)} * \mathbf{u}^{(2)})(\mathbf{x},t) = \dot{\mathbf{u}}^{(1)}(\mathbf{x},t)\cdot\mathbf{u}_0^{(2)}(\mathbf{x})-\mathbf{v}_0^{(1)}(\mathbf{x})\cdot\mathbf{u}^{(2)}(\mathbf{x},t)+(\dot{\mathbf{u}}^{(1)} * \dot{\mathbf{u}}^{(2)})(\mathbf{x},t),$$

$$(\mathbf{J}[\ddot{\boldsymbol{\varphi}}^{(1)}] * \boldsymbol{\varphi}^{(2)})(\mathbf{x},t) = \mathbf{J}(\mathbf{x})[\dot{\boldsymbol{\varphi}}^{(1)}(\mathbf{x},t)]\cdot\boldsymbol{\varphi}_0^{(2)}(\mathbf{x})-\mathbf{J}(\mathbf{x})[\mathbf{w}_0^{(1)}(\mathbf{x})]\cdot\boldsymbol{\varphi}^{(2)}(\mathbf{x},t),$$

$$+(\mathbf{J}[\dot{\boldsymbol{\varphi}}^{(1)}] * \dot{\boldsymbol{\varphi}}^{(2)})(\mathbf{x},t),$$

$$(\bar{\mathbf{b}}^{(1)} * \dot{\mathbf{b}}^{(2)})(\mathbf{x},t) = -\bar{\mathbf{b}}^{(1)}(\mathbf{x},t)\cdot\mathbf{b}_0^{(2)}(\mathbf{x})+(\mathbf{b}^{(1)} * \mathbf{b}^{(2)})(\mathbf{x},t). \tag{48}$$

The properties of the convolution, the symmetry of tensor **J** and eqns (46)–(48) prove eqn (45).

Remark 1

If initial conditions (26) are homogeneous, then the reciprocity relation (45) becomes:

$$\int_B (F^{(1)} * U^{(2)})(\mathbf{x},t)\,\mathrm{d}v+\int_{\partial B}(T^{(1)} * U^{(2)})(\mathbf{x},t)\,\mathrm{d}a$$

$$=\int_B (F^{(2)} * U^{(1)})(\mathbf{x},t)\,\mathrm{d}v+\int_{\partial B}(T^{(2)} * U^{(1)})(\mathbf{x},t)\,\mathrm{d}a. \tag{49}$$

Making use of notations (36), convolutions (39), inner products (37) and (38), relations (5) and the Titchmarsh's theorem, we conclude that eqn (49) can be written in the equivalent form:

$$\int_B \left(\mathbf{X}^{(1)} * \dot{\mathbf{u}}^{(2)} + \mathbf{Y}^{(1)} * \dot{\boldsymbol{\varphi}}^{(2)} - \frac{1}{T_0} r^{(1)} * \boldsymbol{\theta}^{(2)} + \mathbf{j}^{(1)} * \mathbf{e}^{(2)}\right) \mathrm{d}v$$

$$+ \int_{\partial B} \left(\mathbf{s}^{(1)} * \dot{\mathbf{u}}^{(2)} + \mathbf{m}^{(1)} * \dot{\boldsymbol{\varphi}}^{(2)} + \frac{1}{T_0} q^{(1)} * \boldsymbol{\theta}^{(2)} + (\mathbf{b}^{(1)} \times \mathbf{n}) * \mathbf{e}^{(2)}\right) \mathrm{d}a$$

$$= \int_B \left(\mathbf{X}^{(2)} * \dot{\mathbf{u}}^{(1)} + \mathbf{Y}^{(2)} * \dot{\boldsymbol{\varphi}}^{(1)} - \frac{1}{T_0} r^{(2)} * \boldsymbol{\theta}^{(1)} + \mathbf{j}^{(2)} * \mathbf{e}^{(1)}\right) \mathrm{d}v$$

$$+ \int_{\partial B} \left(\mathbf{s}^{(2)} * \dot{\mathbf{u}}^{(1)} + \mathbf{m}^{(2)} * \dot{\boldsymbol{\varphi}}^{(1)} + \frac{1}{T_0} q^{(2)} * \boldsymbol{\theta}^{(1)} + (\mathbf{b}^{(2)} \times \mathbf{n}) * \mathbf{e}^{(1)}\right) \mathrm{d}a. \quad (50)$$

Remark 2

Reciprocity relations (49) and (50) are very useful in deducing an integral representation formula of a solution of the mixed problem.

Theorem 3

If $\varrho^{(\alpha)}$ is a solution of the mixed problem in B corresponding to the external data system $L^{(\alpha)}$, $\alpha = 1, 2$, then the following reciprocity relation holds:

$$\int_B (\tilde{F}^{(1)} * U^{(2)} - i * \mathbf{b}^{(1)} * \mathbf{b}_0^{(2)})(\mathbf{x}, t) \, \mathrm{d}v + \int_{\partial B} (i * T^{(1)} * U^{(2)})(\mathbf{x}, t) \, \mathrm{d}a$$

$$= \int_B (\tilde{F}^{(2)} * U^{(1)} - i * \mathbf{b}^{(2)} * \mathbf{b}_0^{(1)})(\mathbf{x}, t) \, \mathrm{d}v + \int_{\partial B} (i * T^{(2)} * U^{(1)})(\mathbf{x}, t) \, \mathrm{d}a, \quad (51)$$

where

$$\tilde{F}^{(\alpha)} = (\tilde{\mathbf{X}}^{(\alpha)}, \tilde{\mathbf{Y}}^{(\alpha)}, -i * W^{(\alpha)}, i * \bar{\mathbf{j}}^{(\alpha)} - \tfrac{1}{2} t^2 \mathbf{d}_0^{(\alpha)}), \quad \alpha = 1, 2. \quad (52)$$

Proof

Taking the convolution of relation (45) with function i defined by eqn (3), we conclude, with the aid of the properties of convolution and notations (35), that eqn (51) is true.

Remark 3

This new reciprocity relation has the advantage that the initial conditions, excepting that referring to the magnetic flux density, are included in the pseudo-loads (52).

Remark 4

If in eqn (51) the initial conditions are homogeneous, then from the Titchmarch's theorem, we conclude that eqn (51) becomes eqn (49).

Remark 5

The basic result established in Section 5 implies a uniqueness result for the solution of the mixed problem defined in Section 4.

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